The Causal Phase in QED_3

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Abstract

The operator **S** in Fock space which describes the scattering and particle production processes in an external time-dependent electromagnetic potential A can be constructed from the one-particle S-matrix up to a physical phase $\lambda[A]$. In this work we determine this phase for QED in (2+1) dimensions, by means of causality, and show that no ultraviolet divergences arise, in contrast to the usual formalism of QED.

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1. Introduction

The efforts to test quantum electrodynamics in strong electromagnetic fields in the late 70's brought into evidence the external field problem in the context of the spontaneous dacay of the neutral to a charged vacuum through pair creation, in heavy-ion collision experiments. Although the physics of the quantized electron-positron field in interaction with a classical electromagnetic field is well understood, some mathematical aspects of the theory are rather involving, particularly the definition of the scattering operator in Fock

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space for time-dependent external fields (for a review see [1] and references therein).

In this paper we introduce the scattering operator **S** in Fock space for quantum electrodynamics in (2+1) dimensional space-time, in an external time-dependent electromagnetic field A and show that it is unitary and uniquely determined up to a phase. This phase is related to vacuum fluctuations due to the presence of the external potential $A_\mu(x)$ and, therefore, must depend on it. We then determine the phase $\lambda[A]$ in lowest order of perturbation theory, by imposing Bogoliubov's local causality condition on **S**, and show that the vacuum-vacuum amplitude is ultraviolet finite.

The construction of the S-matrix in Fock space is outlined in section 2. In section 3 we present a brief digression on the global as well as the differential causality conditions for the S-operator in Fock space. Section 4 is devoted to the derivation of the causal phase for QED_3 in lowest order of perturbation theory, applying the concepts introduced in the preceeding section, and exploiting the connnection with vacuum polarization. In section 5 we summarize our conclusions.

2. The Scattering Operator in Fock Space

We start from the one-particle Hamiltonian

$$H(t) = H_0 + V(t) \ , \tag{1}$$

where

$$V(t) = e(V(t,\vec{x}) - \vec{\alpha}.\vec{A}(t,\vec{x})) \ . \tag{2}$$

The potentials are assumed to vanish for $t \longrightarrow \pm\infty$ in such a way that the wave operators

$$W_{\substack{in \\ out}} = s - \lim_{t\to\pm\infty} U(t,0)^\dagger e^{-iH_0 t} \tag{3}$$

exist, together with a unitary S-matrix

$$S = W_{out}^\dagger W_{in} \ . \tag{4}$$

Since by assumption we have the free dynamics for $t \longrightarrow \pm\infty$, we settle second quantization on the Fock representation of the free Dirac field

$$\psi(f) = b(P_+^0 f) + d(P_-^0 f)^\dagger \ . \tag{5}$$

Here $P^0_\pm$ are the projection operators on the positive and negative spectral subspaces of the one-particle free Dirac Hamiltonian H_0, respectively.

The second quantized S-matrix in Fock space is now defined by

$$\psi(S^\dagger f) = \mathbf{S}^{-1}\psi(f)\mathbf{S} \ , \tag{6}$$

$$\psi(S^\dagger f)^\dagger = \mathbf{S}^{-1}\psi(f)^\dagger\mathbf{S} \ , \ \forall f \in \mathcal{H}_1 \ , \tag{7}$$

if it exists. We have taken the adjoint $S^\dagger$ in the test functions since $\psi(f)$ is antilinear in f. It follows from the above definitions that $\mathbf{S}$ is unitary and uniquely determined up to a phase. In order to prove this assertion we proceed as in reference [2].

Proposition: $\mathbf{S}$ is uniquely determined by (6) and (7) up to a factor.

Proof. If $\tilde{\mathbf{S}}$ is another operator in $\mathcal{F}$, satisfying (6), then

$$\mathbf{S}^{-1}\psi(f)\mathbf{S} = \tilde{\mathbf{S}}^{-1}\psi(f)\tilde{\mathbf{S}} \ ,$$

$$\tilde{\mathbf{S}}\mathbf{S}^{-1}\psi(f) = \psi(f)\tilde{\mathbf{S}}\mathbf{S}^{-1} \ , \ \forall f \in \mathcal{H}_1 \ ,$$

and the same is true for all $\psi^\dagger(f)$. From the irreducibility of the Fock representation, we have

$$\tilde{\mathbf{S}}\mathbf{S}^{-1} = \alpha\mathbf{1} \ \text{ i.e. } \ \tilde{\mathbf{S}} = \alpha\mathbf{S} \ . \tag{8}$$

$\square$

Now, taking the adjoint of (6)

$$\psi(S^\dagger f)^\dagger = \mathbf{S}^\dagger\psi(f)^\dagger\mathbf{S}^{-1^\dagger} \ ,$$

and comparing with (7), it follows again from the irreducibility of the Fock representation that

$$\mathbf{S}^\dagger = \rho\mathbf{S}^{-1} \ . \tag{9}$$

If we take the adjoint and the inverse of this equation, namely

$$\mathbf{S} = \rho^*\mathbf{S}^{-1^\dagger} \ ,$$

$$\mathbf{S}^{\dagger^{-1}} = \rho^{-1}\mathbf{S} \ ,$$

we find that

$$\rho^* = \rho \ .$$

From (8) and (9) we obtain

$$\tilde{\mathbf{S}}^\dagger = |\alpha|^2 \rho \tilde{\mathbf{S}}^{-1} \ . \tag{10}$$

Therefore, we may choose

$$|\alpha|^2 = \rho^{-1} \tag{11}$$

such that the operator $\tilde{\mathbf{S}}$ becomes unitary. Since the absolute value of α in (8) is fixed by (11), $\tilde{\mathbf{S}}$ is uniquely determined up to a phase $e^{i\lambda}$. However, this phase $\lambda[A]$ is physical because it depends on the external potential $A_\mu(x)$. As we shall see this phase will be fixed by the requirement of causality of $\mathbf{S}$.

The S-matrix $\mathbf{S}$ in Fock space exists, if and only if $P_+ S P_-$ is a Hilbert-Schmidt operator. In this case it is given by

$$\mathbf{S} = C \, e^{S_{+-} S_{--}^{-1} b^\dagger d^\dagger} : e^{(S_{++}^{\dagger -1} - 1) b^\dagger b} :: e^{(1 - S_{--}^{-1}) d d^\dagger} : e^{S_{--}^{-1} S_{-+} d b} \ , \tag{12}$$

where

$$S_{ij} = P_i S P_j \ , \ \ i, j = +, - \tag{13}$$

and

$$|C|^2 = \det(1 - S_{+-} S_{-+}^\dagger) \ . \tag{14}$$

The first factor in (12) describes electron-positron pair creation, the second one electron scattering, the third one positron scattering and the last one pair annihilation.

3. The Condition of Causality

In the one-particle theory the condition that a change in the interaction law in any space-time region can influence the evolution of the system only at subsequent times can be translated into the factorization of the S-matrix

$$S[A] = S_2 S_1 \ , \ \ S_j \overset{def}{=} S[A_j] \ , \tag{15}$$

where we have written the electromagnetic potential as

$$A^\mu(x) = A_1^\mu(x) + A_2^\mu(x) \ , \tag{16}$$

which is the sum of two parts with disjoint supports in time

$$\mathrm{supp}\, A_1 \subset (-\infty, r] \ , \ \ \mathrm{supp}\, A_2 \subset [r, +\infty) \ . \tag{17}$$

A similar factorization should hold from Eq.(6) for the S-operator $\mathbf{S}$ in Fock space,

$$(\Omega, \mathbf{S}\Omega) = (\Omega, \mathbf{S_2 S_1}\Omega) \ . \tag{18}$$

We call (18) global causality condition for the Fock space S-operator in contrast to the differential condition[4]

$$\frac{\delta}{\delta A_\mu(y)} \left(\Omega, \mathbf{S}^\dagger \frac{\delta \mathbf{S}}{\delta A_\nu(x)} \Omega \right) = 0, \ \text{ for } \ x^0 < y^0 \ . \tag{19}$$

We have seen in the last section that the S-matrix in Fock space can be uniquely determined up to a phase,

$$\mathbf{S} = e^{i\varphi} \tilde{\mathbf{S}} \ , \tag{20}$$

where $\tilde{\mathbf{S}}$ is unitary, and given by expression (12). Inserting (20) into (19) we obtain

$$\frac{\delta}{\delta A_\mu(y)} \left(\mathbf{S}\Omega, \frac{\delta \mathbf{S}}{\delta A_\nu(x)} \Omega \right) =$$

$$i \frac{\delta^2 \varphi}{\delta A_\mu(y) \delta A_\nu(x)} + \frac{\delta}{\delta A_\mu(y)} \left(\tilde{\mathbf{S}}\Omega, \frac{\delta \tilde{\mathbf{S}}}{\delta A_\nu(x)} \Omega \right) \ . \tag{21}$$

It can be shown from the unitarity of $\tilde{\mathbf{S}}$ that the last term in (21) is purely imaginary. Consequently, the real part of the causality condition (19) is automatically satisfied while for the imaginary part we may choose φ conveniently such that (19) holds.

4. The Causal Phase

We now turn to the determination of the causal phase in lowest order of perturbation theory. From (6) we have

$$\tilde{\mathbf{S}}\Omega = C(\Omega + \sum_{mn} (S_{+-})_{mn} b_m^\dagger d_n^\dagger \Omega + \ldots) \ , \tag{22}$$

where we have put S_{--}^{-1} equal to the unity in lowest order. Taking the functional derivative of (22) with respect to $A_\nu(x)$ and keeping only terms of order $O(A)$ in the resulting expression, we arrive at

$$\left(\tilde{\mathbf{S}}\Omega, \frac{\delta\tilde{\mathbf{S}}}{\delta A_\nu(x)}\Omega\right) = iC^2\,\Im m \mathrm{Tr}\left(S_{-+}^\dagger \frac{\delta S_{+-}}{\delta A_\nu(x)}\right) \ . \tag{23}$$

In lowest order we may set $C^2 = 1$.

The local causality condition (19) together with expressions (21) and (23) yield

$$F(x,y) \stackrel{def}{=} \frac{\delta^2\varphi}{\delta A_\mu(y)\delta A_\nu(x)} + \Im m \frac{\delta}{\delta A_\mu(y)}\mathrm{Tr}\left((S_{+-})^\dagger \frac{\delta S_{+-}}{\delta A_\nu(x)}\right) = 0 \tag{24}$$

for $x^0 < y^0$.

Next we calculate the second term in (24). In lowest order of perturbation theory, we have

$$S_{+-}^{(1)} = -iP_+(\mathbf{p})\gamma^0 e\not{A}(p+q)P_-(-\mathbf{q}) \ . \tag{25}$$

As in reference [3] we use the following representation for the Dirac matrices in (2+1) dimensions:

$$\gamma^0 = \sigma_3\,, \ \ \gamma^1 = i\sigma_1\,, \ \ \gamma^2 = i\sigma_2 \tag{26}$$

where σ_j are the Pauli matrices.

From (25) we obtain

$$\mathrm{Tr}\frac{\delta}{\delta A_\mu(y)}(S_{+-})^\dagger \frac{\delta S_{+-}}{\delta A_\nu(x)} =$$

$$e^2(2\pi)^{-3}\int d^2p\int d^2q\, e^{i(p+q)(x-y)}\mathrm{tr}[P_-(-\mathbf{q})\gamma^0\gamma^\mu P_+(\mathbf{p})\gamma^0\gamma^\nu P_-(-\mathbf{q})]$$

$$= -\int d^3k\, e^{ik(x-y)}\hat{P}^{\mu\nu}(k) \ , \tag{27}$$

$\hat{P}^{\mu\nu}(k)$ is not but the vacuum polarization tensor in (2+1) dimensions, which is given by[3]

$$\hat{P}^{\mu\nu}(k) = -e^2(2\pi)^{-3}T^{\nu\mu}(k) \ , \tag{28}$$

where

$$
\begin{aligned}
T^{\nu\mu}(k) &= \int d^3p \; \delta(p^2 - m^2)\Theta(p^0)\delta[(k-p)^2 - m^2] \\
&\times \; \Theta(k^0 - p^0)\, t^{\nu\mu}(k,p)
\end{aligned}
\tag{29}
$$

with

$$
t^{\nu\mu}(k,p) = \mathrm{tr}[\gamma^\mu(\not{p} + m)\gamma^\nu(\not{k} - \not{p} - m)] \; . \tag{30}
$$

It follows from the gauge invariance of (28) that

$$
\hat{P}^{\mu\nu}(k) = \hat{P}^{\mu\nu}_S(k) + \hat{P}^{\mu\nu}_A(k) \tag{31}
$$

with

$$
\hat{P}^{\mu\nu}_S(k) = (k^\mu k^\nu - k^2 g^{\mu\nu})\tilde{B}(k^2) \; , \tag{32}
$$

$$
\hat{P}^{\mu\nu}_A(k) = i m \varepsilon^{\mu\nu\alpha} k_\alpha \tilde{\Pi}^{(2)}(k^2) \; . \tag{33}
$$

Performing the trace in (30) and the resulting momentum integral in (29) we find that[3]

$$
\tilde{B}(k^2) = \frac{-e^2}{2(4\pi)^2}\frac{k^2 + 4m^2}{k^2}\Theta(k^2 - 4m^2)\frac{\Theta(k_0)}{\sqrt{k^2}} \; , \tag{34}
$$

$$
\tilde{\Pi}^{(2)}(k^2) = \frac{-e^2}{2(2\pi)^2}\Theta(k^2 - 4m^2)\frac{\Theta(k_0)}{\sqrt{k^2}} \; . \tag{35}
$$

Substituting (27) and (28) in (24), we rewrite the causal function $F(x,y)$ as

$$
F(x,y) = \frac{\delta^2\varphi}{\delta A_\mu(y)\delta A_\nu(x)} + \frac{e^2}{(2\pi)^3}\Im m \int d^3k \, e^{ik(x-y)} T^{\nu\mu}(k) \; . \tag{36}
$$

We can evaluate the imaginary part of the last term in the above equation taking into account (28) and (31)-(35). Thus, we have

$$
T^{\nu\mu}(k) = T^{\nu\mu}_S(k) + T^{\nu\mu}_A(k) \; , \tag{37}
$$

where $T^{\nu\mu}_S(k)$ is real and even in k while $T^{\nu\mu}_A(k)$ is imaginary and odd in k. Hence,

$$F(x,y) = \frac{\delta^2\varphi}{\delta A_\mu(y)\delta A_\nu(x)} +$$
$$\frac{e^2}{(2\pi)^3}\left[\int_{k_0>0} d^3k\, \sin k(x-y) T_S^{\nu\mu}(k) - i\int_{k_0>0} d^3k\, \cos k(x-y) T_A^{\nu\mu}(k)\right] . \tag{38}$$

In order to write the last term in (38) as a complex Fourier transform we must continue $T^{\nu\mu}(k)$ antisymmetrically to $k_0 < 0$

$$F(x,y) = \frac{\delta^2\varphi}{\delta A_\mu(y)\delta A_\nu(x)} + \frac{i}{2}\int d^3k\, e^{-ik(x-y)}[d_S^{\mu\nu}(k) - d_A^{\mu\nu}(k)] , \tag{39}$$

where

$$d_S^{\mu\nu}(k) = (k^\mu k^\nu - k^2 g^{\mu\nu})B(k^2) , \tag{40}$$

$$d_A^{\mu\nu}(k) = im\varepsilon^{\mu\nu\alpha}k_\alpha \Pi^{(2)}(k^2) . \tag{41}$$

and

$$B(k^2) = \frac{-e^2}{2(4\pi)^2}\frac{k^2+4m^2}{k^2}\Theta(k^2-4m^2)\frac{\mathrm{sgn}(k_0)}{\sqrt{k^2}} , \tag{42}$$

$$\Pi^{(2)}(k^2) = \frac{-e^2}{2(2\pi)^2}\Theta(k^2-4m^2)\frac{\mathrm{sgn}(k_0)}{\sqrt{k^2}} . \tag{43}$$

The Fourier transform of a causal function vanishing for $x^0 - y^0 = t < 0$ satisfies a dispersion relation. Since $d_S^{\mu\nu}(k)$ and $d_A^{\mu\nu}(k)$ are real and purely imaginary, respectively, they cannot be the Fourier transform of a causal function. The lacking imaginary part of $d_S^{\mu\nu}(k)$ and the lacking real part of $d_A^{\mu\nu}(k)$ must de supplied by the first term containing the phase $\varphi[A]$,

$$\frac{\delta^2\varphi}{\delta A_\mu(y)\delta A_\nu(x)} = \frac{i}{2}\int d^3k\, e^{-ik(x-y)}[ir_S^{\mu\nu}(k) - r_A^{\mu\nu}(k)] , \tag{44}$$

where

$$r_S^{\mu\nu}(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt \frac{d_S^{\mu\nu}(kt)}{(t-i0)^2(1-t+i0)}$$

$$= \frac{\beta}{2\pi}(k^\mu k^\nu - k^2 g^{\mu\nu}) \left[\frac{1}{\sqrt{k^2}} \left(1 + \frac{4m^2}{k^2}\right) \log\left(\frac{1 - \sqrt{\frac{4m^2}{k^2}}}{1 + \sqrt{\frac{4m^2}{k^2}}} \right) + \frac{4m}{k^2} \right] , \quad (45)$$

and

$$r_A^{\mu\nu}(k) = -\frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{d_A^{\mu\nu}(kt)}{(t - i0)(1 - t + i0)}$$

$$= \frac{i\beta}{2\pi} 4im\varepsilon^{\mu\nu\alpha} \frac{k_\alpha}{\sqrt{k^2}} \log\left(\frac{1 - \sqrt{\frac{4m^2}{k^2}}}{1 + \sqrt{\frac{4m^2}{k^2}}} \right) , \quad (46)$$

with $\beta \equiv -e^2/[2(4\pi)^2]$.

The causal phase is obtained by two integrations

$$\varphi[A] = \frac{1}{2} \int d^3x \int d^3y \, \frac{\delta^2 \varphi}{\delta A_\mu(y) \delta A_\nu(x)} A_\mu(y) A_\nu(x) \; + O(A^4)$$

$$= -\pi^2 \beta \int d^3k \left[\left(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu} \right) \Pi_1^{(1)}(k) + im\varepsilon^{\mu\nu\alpha} k_\alpha \Pi_1^{(2)}(k) \right] A_\mu(k) A_\nu^*(k) , \quad (47)$$

where

$$\Pi_1^{(1)}(k) = \sqrt{k^2} \left(1 + \frac{4m^2}{k^2}\right) \log\left(\frac{1 - \sqrt{\frac{4m^2}{k^2}}}{1 + \sqrt{\frac{4m^2}{k^2}}} \right) + 4m , \quad (48)$$

$$\Pi_1^{(2)}(k) = -\frac{4}{\sqrt{k^2}} \log\left(\frac{1 - \sqrt{\frac{4m^2}{k^2}}}{1 + \sqrt{\frac{4m^2}{k^2}}} \right) . \quad (49)$$

If we decompose the electromagnetic fields which appear in the integrand of (47) into the respective real and imaginary parts we see that $\varphi[A]$ is indeed real. The S-operator in Fock space $\mathbf{S}[A]$ is then completely determined.

By means of (12) and (20) we obtain the vacuum-vacuum amplitude

$$(\Omega, \mathbf{S}\Omega) = Ce^{i\varphi}(\Omega, e^{S_{+-}S_{--}^{-1} b^\dagger d^\dagger} \Omega) = Ce^{i\varphi} . \quad (50)$$

The absolute square

$$|(\Omega, \mathbf{S}\Omega)|^2 = C^2 = 1 - P \quad (51)$$

must be equal to one minus the total probability P of pair creation,

$$P = (2\pi)^2 \int d^3k \, \hat{P}^{\mu\nu}(k) A_\mu(k) A_\nu^*(k) \ , \tag{52}$$

since the external field can change the vaccum state only into pair states. In order to combine the normalization constant C with $e^{i\varphi}$ we write the former in the exponential form

$$C = \exp\left[2\pi^2 \int d^3k \ \ldots \ + O[A^4] \right] \ .$$

Hence, from (31)-(35) we get

$$C = \exp\{-\pi^2\beta \int d^3k \ \left[\left(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu} \right) \Pi_2^{(1)}(k^2) + im\varepsilon^{\mu\nu\alpha} k_\alpha \Pi_2^{(2)}(k^2) \right] \times A_\mu(k) A_\nu^*(k)\} \ , \tag{53}$$

where

$$\Pi_2^{(1)}(k^2) = 2\sqrt{k^2} \left(1 + \frac{4m^2}{k^2} \right) \Theta(k^2 - 4m^2) \ , \tag{54}$$

$$\Pi_2^{(2)}(k^2) = \frac{8\Theta(k^2 - 4m^2)}{\sqrt{k^2}} \ . \tag{55}$$

Finally, taking into account (47)-(49),(50) and (53)-(55), we obtain the vacuum-vacuum amplitude

$$(\Omega, \mathbf{S}\Omega) = \exp\{-i\pi^2\beta \int d^3k \ \left[\left(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu} \right) \Pi^{(1)}(k^2) + im\varepsilon^{\mu\nu\alpha} k_\alpha \Pi^{(2)}(k^2) \right] \times A_\mu(k) A_\nu^*(k)\} \ , \tag{56}$$

where

$$\begin{aligned} \Pi^{(1)}(k^2) &= \Pi_1^{(1)}(k^2) - i\Pi_2^{(1)}(k^2) \ , \\ \Pi^{(2)}(k^2) &= \Pi_1^{(2)}(k^2) - i\Pi_2^{(2)}(k^2) \ . \end{aligned} \tag{57}$$

5. Concluding Remarks

We have considered QED_3 in the presence of an external electromagnetic field A and shown that a unitary scattering operator $\mathbf{S}$ which satisfies the local causality condition can be constructed in Fock space. We have also derived the vacuum-vacuum amplitude and stablished the connection with vacuum polarization in lowest order of perturbation theory. In contrast with the four-dimensional case, the vacuum-vacuum amplitude is ultraviolet finite and exhibits an additional contribution from the antisymmetric part of the vacuum polarization tensor in (2+1)-dimensional space-time[3], which emerges from the topological structure of the theory.

6. References

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